

WEIGHTED SUMS OF THE SQUARES OF THE DISTANCES OF A POINT TO THE SIDELINES OF A TRIANGLE

GEORGI GANCHEV AND NIKOLAI NIKOLOV

ABSTRACT. We study a function, which is a weighted sum of the squares of the distances of an arbitrary point to the sidelines of a triangle. The given weights, considered as barycentric coordinates, determine a point M . We prove that the function reaches its minimum (maximum) at a point, which is isogonal conjugate to M .

1. INTRODUCTION

As usual, we denote by a, b, c the sides of a given $\triangle ABC$ and by S its area. The positive orientation of the plane is determined by $\triangle ABC$.

Let x, y, z be trilinear coordinates of a point with respect to $\triangle ABC$.

It is well known that the Lemoine point K minimizes the sum $x^2 + y^2 + z^2$, i.e. [2]

$$x^2 + y^2 + z^2 \geq \frac{4S^2}{a^2 + b^2 + c^2}.$$

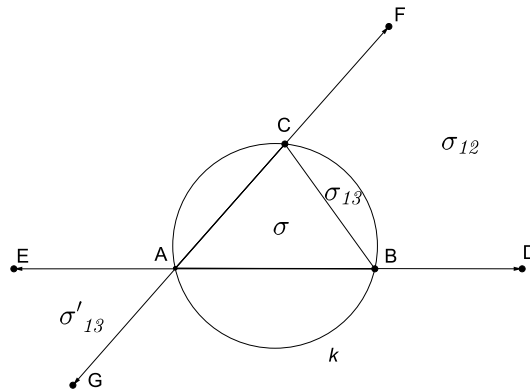
Recently Kimberling [1] obtained several inequalities for the power sums $x^q + y^q + z^q$.

In this note for an arbitrary point X in the plane of $\triangle ABC$ we study a weighted sum $F(X)$ of the squares of the distances of X to the sidelines of the triangle. We give a geometric interpretation of the minimum (maximum) of the function $F(X)$.

2. PRELIMINARIES

Let us recall some properties of isogonal conjugate points with respect to a given $\triangle ABC$.

Given the basic $\triangle ABC$ and its circumcircle $k(ABC)$. Denote by ι the isogonal conjugation with respect to the triangle. The action of ι in the domains (with respect to the vertex A) (Fig. 1) is as follows:



2000 *Mathematics Subject Classification.* Primary 51M04, Secondary 51M16.

Key words and phrases. Weighted sum of the squares of the distances, isogonal conjugate points, barycentric coordinates.

Fig. 1

1) $\iota(\sigma) = \sigma$.

If M is a point on the side BC , then $\iota(M) = A$.

2) $\iota(\sigma_{12}) = \sigma_{12}$.

For any point M on the ray BD^{\rightarrow} we have $\iota(M) = C$; for any point $M \in CF^{\rightarrow}$ $\iota(M) = B$.

3) $\iota(\sigma_{13}) = \sigma'_{13}$, $\iota(\sigma'_{13}) = \sigma_{13}$.

For any point $M \in AE^{\rightarrow}$ $\iota(M) = C$; for any point $M \in AG^{\rightarrow}$ $\iota(M) = B$.

The transformation ι can also be defined for points on the circumcircle k , different from the vertices of the triangle. If M is a point on the arc \widehat{BC} , then $\iota(M)$ is the point at infinity of the line, which is symmetric to the line AM with respect to the bisector of $\angle BAC$.

For an arbitrary point X in the plane of $\triangle ABC$ we denote by x, y, z , the directed distances of the point X to the lines BC, CA, AB , respectively. Then (x, y, z) is the triple of trilinear coordinates of X with respect to the basic triangle. The trilinear coordinates satisfy the equality $ax + by + cz = 2S$. Further we denote by S_1, S_2, S_3 the oriented areas of the triangles BCX, CAX, ABX , respectively. Then $\lambda = \frac{S_1}{S}, \mu = \frac{S_2}{S}, \nu = \frac{S_3}{S}$; $\lambda + \mu + \nu = 1$ are the barycentric coordinates of X with respect to the $\triangle ABC$. The relation between the trilinear coordinates and the barycentric coordinates of the point X is given by

$$\lambda = \frac{ax}{2S}, \quad \mu = \frac{by}{2S}, \quad \nu = \frac{cz}{2S}.$$

We also consider the function $J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} = \left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \right) 2S$, which is defined for all points that do not lie on the sidelines of $\triangle ABC$.

The following characterization of the points in the $\angle BAC$ out of $\triangle ABC$ is useful.

Lemma 1. Let $M(x, y, z)$ be with trilinear coordinates satisfying the conditions $x < 0, y > 0, z > 0$, i.e. M is in the $\angle BAC$ out of $\triangle ABC$. Then

- 1) $M \in \sigma_{12}$ iff $J > 0$;
- 2) M lies on the arc \widehat{BC} iff $J = 0$;
- 3) $M \in \sigma_{13}$ iff $J < 0$.

We also need the following statement.

Lemma 2. Let $M(x, y, z)$ be with trilinear coordinates satisfying the conditions $x > 0, y < 0, z < 0$, i.e. $M \in \sigma'_{13}$. Then $J < 0$.

The isogonal conjugation with respect to barycentric coordinates

Let (λ, μ, ν) , $\lambda + \mu + \nu = 1$ be the barycentric coordinates of a point M , which does not lie on the lines AB, BC, CA or on the circumcircle $k(ABC)$. If (λ', μ', ν') are the barycentric coordinates of the point N , isogonal conjugate to M , then

$$(2.1) \quad \lambda' = \frac{a^2}{\lambda J}, \quad \mu' = \frac{b^2}{\mu J}, \quad \nu' = \frac{c^2}{\nu J}.$$

If (λ, μ, ν) are the barycentric coordinates of a point M , it is useful to consider the *homogeneous* barycentric coordinates of M :

$$(\rho\lambda, \rho\mu, \rho\nu), \quad \rho \neq 0.$$

Then the formulas

$$(2.2) \quad \lambda' = \frac{a^2}{\lambda}, \quad \mu' = \frac{b^2}{\mu}, \quad \nu' = \frac{c^2}{\nu}$$

represent the isogonal conjugation even on the arcs \widehat{BC} , \widehat{CA} or \widehat{AB} of k . If $M(\lambda, \mu, \nu)$ lies on the arc \widehat{BC} , then the point $\iota(M) = N(\lambda', \mu', \nu')$ satisfies the condition $\lambda' + \mu' + \nu' = 0$ and lies on the line at infinity.

A general formulation of the problem

Now, let $(\lambda, \mu, \nu) \neq (0, 0, 0)$ be a triple of fixed real numbers. For any point X with trilinear coordinates (x, y, z) consider the function

$$F(X) = \lambda x^2 + \mu y^2 + \nu z^2,$$

which is a weighted sum of the squares of the directed distances (x, y, z) .

Our aim is to investigate the minima and maxima of the above function.

Further we consider three essential cases:

1. $\lambda\mu\nu \neq 0, \quad \lambda + \mu + \nu > 0;$
2. $\lambda\mu\nu \neq 0, \quad \lambda + \mu + \nu < 0;$
3. $\lambda\mu\nu \neq 0, \quad \lambda + \mu + \nu = 0.$

3. WEIGHTED SUM WITH $\lambda\mu\nu \neq 0, \quad \lambda + \mu + \nu > 0$

Obviously both functions

$$\lambda x^2 + \mu y^2 + \nu z^2, \quad \frac{\lambda x^2 + \mu y^2 + \nu z^2}{\lambda + \mu + \nu}$$

have minima and maxima at the same points. Without loss of generality we can assume that $\lambda + \mu + \nu = 1$.

Thus the problem in this section is to find the minimum (maximum) of the function

$$(3.1) \quad F(X) = \lambda x^2 + \mu y^2 + \nu z^2, \quad \lambda + \mu + \nu = 1.$$

First we consider the case

1.1. $\lambda > 0, \mu > 0, \nu > 0.$

Problem 1. (i) Find the point N that minimizes the function $F(X)$.

(ii) If M is the point with barycentric coordinates (λ, μ, ν) , prove that M and N are isogonal conjugate.

Solution. To solve (i), consider the system

$$(3.2) \quad \begin{aligned} F(X) &= \lambda x^2 + \mu y^2 + \nu z^2, \\ ax + by + cz &= 2S; \end{aligned} \quad x, y, z \in \mathbb{R}$$

and interpret (x, y, z) as Cartesian coordinates in the three dimensional Euclidean space. The level surfaces of the function $F(X)$ are the ellipsoids

$$\varepsilon(k): \quad \lambda x^2 + \mu y^2 + \nu z^2 = k, \quad k = \text{const} \in (0, \infty).$$

Geometrically, to find the point that minimizes the function $F(X)$ in (2.2), means to find k so that the ellipsoid $\varepsilon(k)$ is tangent to the plane $\pi: ax + by + cz = 2S$ and then to determine the touch-point N of $\varepsilon(k)$ to π .

The tangent plane to $\varepsilon(k)$ at a point (x_0, y_0, z_0) is given by the equality

$$\tau : \lambda x_0 x + \mu y_0 y + \nu z_0 z = k.$$

Then the condition $\tau \equiv \pi$ implies that

$$(3.3) \quad \begin{aligned} \frac{\lambda x_0}{a} &= \frac{\mu y_0}{b} = \frac{\nu z_0}{c} = t, \\ t(ax_0 + by_0 + cz_0) &= k, \\ ax_0 + by_0 + cz_0 &= 2S. \end{aligned}$$

Solving (3.3), we find:

$$t = \frac{2S}{J}, \quad k = \frac{4S^2}{J}, \quad \left(J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} \right)$$

and

$$(3.4) \quad x_0 = \frac{2S}{J} \frac{a}{\lambda}, \quad y_0 = \frac{2S}{J} \frac{b}{\mu}, \quad z_0 = \frac{2S}{J} \frac{c}{\nu}.$$

Now, taking into account (3.4), we conclude that the point N minimizing the function $F(X)$ has barycentric coordinates (λ', μ', ν') with respect to $\triangle ABC$ given by

$$(3.5) \quad \lambda' = \frac{ax_0}{2S} = \frac{a^2}{\lambda J}, \quad \mu' = \frac{by_0}{2S} = \frac{b^2}{\mu J}, \quad \nu' = \frac{cz_0}{2S} = \frac{c^2}{\nu J}$$

and $F_{min} = k = \frac{4S^2}{J}$, which solves (i).

To prove (ii), let us denote by M the point with barycentric coordinates (λ, μ, ν) . Comparing with (1.1) we conclude that formulas (3.5) are a representation of the isogonal conjugation in barycentric coordinates. Hence, the point $N(\lambda', \mu', \nu')$ is the isogonal conjugate one to the point $M(\lambda, \mu, \nu)$. \square

In this case M and N are in σ .

Next we consider the case

1.2. $\lambda < 0, \mu > 0, \nu > 0$.

In this case the problem states as follows:

Problem 2. Prove that $F(X)$ has a minimum if and only if

$$J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} < 0.$$

(i) Find the point N that minimizes the function $F(X)$.

(ii) If M is the point with homogeneous barycentric coordinates (λ, μ, ν) , prove that M and N are isogonal conjugate.

Solution. Consider the system (3.2). In this case any level surface of the function $F(X)$

$$\varepsilon(k) : \lambda x^2 + \mu y^2 + \nu z^2 = k, \quad k = \text{const} \in \mathbb{R}$$

is one of the following: one sheet hyperboloid if $k > 0$; cone if $k = 0$; two sheet hyperboloid if $k < 0$.

Geometrically, to find the point that minimizes the function $F(X)$ in (3.2), means to find $k < 0$ so that the two sheet hyperboloid $\varepsilon(k)$ is tangent to the plane $\pi : ax + by + cz = 2S$ and then to determine the touch-point N of $\varepsilon(k)$ to π .

The plane π can be tangent to $\varepsilon(k)$ only if $k < 0$.

Similarly to the solution of Problem 1 we obtain the system (3.4), which implies that

$$tJ = 2S, \quad 2St = k.$$

Therefore $\varepsilon(k)$ is tangent to the plane π only in the case $J < 0$.

Further, we find the coordinates of the touch-point

$$(3.6) \quad x_0 = \frac{2S}{J} \frac{a}{\lambda}, \quad y_0 = \frac{2S}{J} \frac{b}{\mu}, \quad z_0 = \frac{2S}{J} \frac{c}{\nu}.$$

Thus, the point N minimizing the function $F(X)$ lies in the domain σ'_{13} .

Let M be the point with barycentric coordinates (λ, μ, ν) . Then the formulas (3.6) show that the point $N(\lambda', \mu', \nu')$ has barycentric coordinates

$$\lambda' = \frac{a^2}{\lambda J}, \quad \mu' = \frac{b^2}{\mu J}, \quad \nu' = \frac{c^2}{\nu J}$$

and it is isogonal conjugate to the point M .

Hence, the triple (λ, μ, ν) determines a point M in the domain σ_{13} and

$$F_{min} = F(N) = \frac{4S^2}{J} < 0,$$

where $N \in \sigma'_{13}$ is the isogonal conjugate to the point M . □

1.3. $\lambda > 0, \mu < 0, \nu < 0$.

In this case the problem states as follows:

Problem 3. Prove that the function $F(X)$ has neither minimum, nor maximum.

Solution. Let M be the point with barycentric coordinates (λ, μ, ν) . Then $M \in \sigma'_{13}$ and $J < 0$. Similarly to the case 1.2 we obtain that in our case any level surface of the function $F(X)$

$$\varepsilon(k): \quad \lambda x^2 + \mu y^2 + \nu z^2 = k, \quad k = \text{const} \in \mathbb{R}$$

is one of the following: one sheet hyperboloid if $k < 0$; cone if $k = 0$; two sheet hyperboloid if $k > 0$. Since $J < 0$, then $k = \frac{4S^2}{J} < 0$ and π can not be a tangent plane to any two sheet hyperboloid. Hence the function $F(X)$ has neither minimum nor maximum. □

4. WEIGHTED SUM WITH $\lambda\mu\nu \neq 0, \lambda + \mu + \nu < 0$

If $\lambda + \mu + \nu < 0$, then we put $\bar{\lambda} = -\lambda, \bar{\mu} = -\mu, \bar{\nu} = -\nu$ and consider the function

$$\bar{F}(X) = \bar{\lambda}x^2 + \bar{\mu}y^2 + \bar{\nu}z^2 = -F(X), \quad \bar{\lambda} + \bar{\mu} + \bar{\nu} > 0.$$

We suppose again that $\bar{\lambda} + \bar{\mu} + \bar{\nu} = 1$ and consider the point M with barycentric coordinates $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$.

Comparing with Section 2 we have the following.

2.1. $\lambda < 0, \mu < 0, \nu < 0$

Under these conditions $J = \frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} < 0$.

Then

$$M(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \sigma, \quad N = \iota(M) \in \sigma, \quad \bar{J} > 0, \quad F_{max} = F(N) = \frac{4S^2}{J} < 0.$$

2.2. $\lambda > 0, \mu < 0, \nu < 0$

2.2.1. $J < 0$

The function $\bar{F} = -F$ has no minimum or maximum.

2.2.2 $J > 0$

$$M(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \sigma_{13}, N = \iota(M) \in \sigma'_{13}, \bar{J} < 0, \quad F_{max} = F(N) = \frac{4S^2}{J} > 0.$$

2.3. $\lambda < 0, \mu > 0, \nu > 0$

The function $\bar{F} = -F$ has no minimum or maximum.

5. WEIGHTED SUM WITH $\lambda\mu\nu \neq 0, \lambda + \mu + \nu = 0$

Let us consider the system of mass points $\{A(\lambda), B(\mu), C(\nu)\}$ and denote by $P(\mu + \nu)$ the center of mass of the system $\{B(\mu), C(\nu)\}$ (Fig. 2). If O is an arbitrary point, we have

$$\lambda \overrightarrow{OA} + \mu \overrightarrow{OB} + \nu \overrightarrow{OC} = -\lambda \overrightarrow{AP} = \mu \overrightarrow{AB} + \nu \overrightarrow{AC} = \overrightarrow{v} = \text{const.}$$

Calculating

$$\overrightarrow{v}^2 = -\lambda\mu\nu \left(\frac{a^2}{\lambda} + \frac{b^2}{\mu} + \frac{c^2}{\nu} \right) = -\lambda\mu\nu J,$$

we obtain the geometric meaning of J in the case $\lambda + \mu + \nu = 0$.

The condition $\overrightarrow{v}^2 > 0$ implies that

$$(5.1) \quad \lambda\mu\nu J < 0.$$

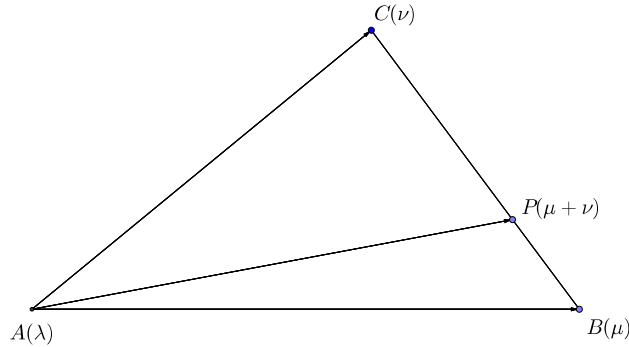


Fig. 2

In this section we consider two cases: $(\lambda < 0, \mu > 0, \nu > 0)$ and $(\lambda > 0, \mu < 0, \nu < 0)$.

3.1. $\lambda\mu\nu < 0$.

In this case $M(\lambda, \mu, \nu)$ can be interpreted as a point at infinity, i.e. the point at infinity of the line AP (Fig. 2). The inequality (5.1) implies that $J > 0$. Using similar arguments as in Section 2 we conclude that the function $F(X)$ has neither minimum nor maximum.

3.2. $\lambda\mu\nu > 0$.

Under these conditions the inequality (5.1) implies that $J < 0$. We consider the function $\bar{F} = -F$. Since $\bar{\lambda}\bar{\mu}\bar{\nu} < 0$, comparing with the previous case we conclude that \bar{F} i.e. F has no minimum or maximum.

6. WEIGHTED SUM WITH ONE OR TWO ZERO WEIGHTS

4.1. $\lambda = 0, \mu > 0, \nu > 0$.

In this case the level surfaces of the function $F(X)$ are the elliptic cylinders

$$\mu y^2 + \nu z^2 = k, \quad k = \text{const} \in (0, \infty),$$

and the axis Ox , when $k = 0$. The level surfaces intersect the plane $ax + by + cz = 2S$ into ellipses by $k > 0$ and in the point $(2S, 0, 0)$ by $k = 0$.

The minimum of the function $F(X)$ is $F_{min} = 0$ and it occurs when $y = z = 0$.

Thus $M(0, \mu, \nu)$ is a point on the sideline BC and the minimum $F_{min} = 0$ occurs when $N \equiv A$, i.e. M and N are again isogonal conjugate.

4.2. $\lambda = 0, \mu < 0, \nu < 0$.

$$M(0, \bar{\mu}, \bar{\nu}) \in BC, N = \iota(M) = A, F_{max} = F(N) = 0.$$

4.3. $\lambda = 0, \mu\nu < 0$.

In this case the level surfaces of the function $F(X)$

$$\mu y^2 + \nu z^2 = k, \quad k \in (-\infty, +\infty)$$

are hyperbolic cylinders if $k \neq 0$ and two planes if $k = 0$.

Hence the function $F(X)$ has no minimum or maximum.

5.1. $\lambda = 1, \mu = \nu = 0$.

It is clear that $F_{min} = 0$ and it occurs when $x = 0$.

Thus $M(1, 0, 0) \equiv A$ and N is any point on BC , i.e. M and N are again isogonal conjugate.

5.2. $\lambda = -1, \mu = 0, \nu = 0$.

$$M(1, 0, 0) \equiv A, N \in BC, F_{max} = F(N) = 0.$$

Summarizing we get the following:

1. $\lambda + \mu + \nu > 0$.

If $M(\lambda, \mu, \nu)$ is an inner point for the circumcircle k , or coincides with a vertex of $\triangle ABC$, then F has a minimum and $F_{min} = F(N)$, $N = \iota(M)$.

2. $\lambda + \mu + \nu < 0$.

If $M(-\lambda, -\mu, -\nu)$ is an inner point for the circumcircle k , or coincides with a vertex of $\triangle ABC$, then F has a maximum and $F_{max} = F(N)$, $N = \iota(M)$.

3. $\lambda + \mu + \nu = 0$.

In this case the function F has neither a minimum nor a maximum.

7. EXAMPLES

We choose as a typical example the following pair of conjugate points: the circumcenter O and the orthocenter H .

1. Let $M \equiv O$. Then $\lambda = \sin 2\alpha$, $\mu = \sin 2\beta$, $\nu = \sin 2\gamma$ and the point O generates the function

$$F(X) = \sin 2\alpha x^2 + \sin 2\beta y^2 + \sin 2\gamma z^2.$$

1.1. $\triangle ABC$ is acute-angled and $O \in \sigma$. Simple calculations show that

$$J = \frac{4S}{\cos \alpha \cos \beta \cos \gamma}, \quad F_{\min} = F(H) = 4S \cos \alpha \cos \beta \cos \gamma.$$

1.2. $\alpha = 90^\circ$ and O is the midpoint of BC . Then

$$F_{\min} = F(A) = 0.$$

1.3. $\alpha > 90^\circ$ and $O \in \sigma_{13}$. Then

$$J = \frac{4S}{\cos \alpha \cos \beta \cos \gamma} < 0, \quad F_{\min} = F(H) = 4S \cos \alpha \cos \beta \cos \gamma < 0.$$

Remark. *The function $F(X)$ generates geometric inequalities.*

Let $M(\lambda, \mu, \nu) \in \sigma$ generate the function (3.1). If we choose a concrete triangle center X in σ with trilinear coordinates (x, y, z) and replace in (3.1), then we obtain the geometric inequality

$$F(X) \geq F_{\min} = \frac{4S^2}{J}.$$

The equality occurs if and only if $X \equiv \iota(M)$.

In a similar way the case $M \in \sigma_{13}$ generates even more interesting geometric inequalities.

REFERENCES

- [1] C. Kimberling, *Trilinear Distance Inequalities for the Symmedian Point, the Centroid, and Other Triangle Centers*, Forum Geometricorum, 10 (2010), 135-139.
- [2] E. Lemoine, *Sur quelques propriétés d'un point remarquable d'un triangle*, Association française pour l'avancement des sciences, Congrès (002; 1873; Lyon), (1874), 90-95.

BULGARIAN ACADEMY OF SCIENCES, INSTITUTE OF MATHEMATICS AND INFORMATICS, ACAD. G. BONCHEV
STR. BL. 8, 1113 SOFIA, BULGARIA

E-mail address: ganchev@math.bas.bg

E-mail address: nik@math.bas.bg